



LIMITING FORM OF EXPONENTIAL TIME DIFFERENCING SCHEMES

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ABSTRACT

The problem of finding a stable numerical scheme that handles stiff ordinary differential equations has been around for decades and several methods of solution have been developed over the years. The Exponential Time Differencing (ETD) developed by Cox and Matthew (2002) is one of the best schemes developed ever, though it is yet to receive a wide appreciation. The purpose of this paper is to show that the Exponential Time Differencing Schemes of orders 1, 2, 3 and 4 can be modified and simplified to Adams Bashforth Schemes of the corresponding orders as the stiff coefficients gets smaller.

Keywords: stiff systems, exponential time differencing, Adams Bashforth, Limiting Form.

INTRODUCTION

There are two widely used methods for discretising the spatial derivatives of a partial differential equation (PDE) and they are finite difference methods and spectral method. Stiffness is often observed when a spectral method is adopted. By stiffness, we mean that a part of the solution evolves more rapidly while the other part evolves very slowly and thereby increasing the error rapidly.

Although, there has been no particular definition for the phenomenon called stiffness, there are some pointers that give a clue about the stiffness of a system of ordinary differential equations (ODE). Stiffness is common with a system of ODEs whose eigenvalues vary widely (i.e. the ratio of the highest eigenvalue to the smallest eigenvalue is high). A system whose stability depends on the choice of step size instead of accuracy is said to be stiff. If adaptive methods do not converge

or take a very long time to converge, or if two different methods do not agree on the solution then the system of ODE is probably stiff.

This phenomenon is common in practical problems. Curtiss and Hirschfelder (1952) show the existence of stiffness in chemical kinetics and electrical circuits.

Several methods of solution have been developed over the years, some of which are the forward interpolation process (Curtiss and Hirschfelder, 1952), integrating factor (Lawson, 1967), exact treatment of linear part (Beylkin et al., 1998), Yee algorithm (Yee, 1966) and Holland (1994).

This paper is focused on the class of numerical methods developed in Cox and Matthews (2002) and modified by Kassam and Trefethen (2005). Several works on ETD include Gentian Zavalani (2014), F. de la Hoz, F. Vadillo (2009)

The emphasis in this paper is on the class of methods developed in Cox and Matthews (2002). The problem of the form

$$\frac{dy}{dx} = cy + f(x, y) \tag{1}$$

is solved using the integrating factor and the class of ETD is derived. This paper obtains the limiting form of the first four ETD schemes as $c \rightarrow 0$ and it is observed that the different ETD schemes reduces to some known methods.

EXPONENTIAL TIME DIFFERENCING

The class of the exponential time differencing scheme is derived by considering the stiff ordinary differential equation of the form

$$\frac{dy}{dx} = cy + f(x, y) \tag{2}$$

and then using the method of integrating factor, we have

$$\frac{d}{dx}(ye^{-cx}) = f(x, y)e^{-cx} \tag{3}$$

so that on integrating in a small interval $[x_{n+1}, x_n]$

$$y_{n+1}e^{-cx_{n+1}} = y_n e^{-cx_n} + \int_{x_n}^{x_{n+1}} f(x, y)e^{-cx} dx \tag{4}$$

and then

$$y_{n+1} = y_n e^{c(x_{n+1}-x_n)} + e^{cx_{n+1}} \int_{x_n}^{x_{n+1}} f(x, y)e^{-cx} dx. \tag{5}$$

Now, put

$$h = x_{n+1} - x_n \tag{6}$$

then,

$$y_{n+1} = y_n e^{ch} + e^{cx_{n+1}} \int_{x_n}^{x_{n+1}} f(x, y)e^{-cx} dx. \tag{7}$$

and put

$$t = x - x_n \tag{8}$$

and so

$$y_{n+1} = y_n e^{ch} - e^{ch} \int_0^h f(t + x_n, y(t + x_n))e^{-ct} dt. \tag{9}$$

Using the Newton backward difference interpolating polynomial, Cox and Matthews (2002) obtained an interpolating polynomial for $f(t + x_n, y(t + x_n))$. Depending on the degree of the polynomial, a class of numerical scheme called the Exponential Time Differencing schemes (ETD) was derived and the result is hereby stated

$$y_{n+1} = y_n e^{ch} + h \sum_{m=0}^{s-1} \left(g_m \sum_{k=0}^m (-1)^k \binom{m}{k} F_{n-k} \right), \tag{10}$$

$$\text{where } g_m = \frac{1}{ch} \left(-1 + \sum_{k=0}^{m-1} \frac{g_k}{m-k} \right). \tag{11}$$

Table 1 shows a list of the ETD schemes of orders 1, 2, 3, and 4. For the derivation of the schemes, refer to Cox and Matthews (2002).

Limiting Form of Exponential Time Differencing Schemes

ORDER	SCHEME	
ETD1	$y_{n+1} = y_n e^{ch} + hf_n g_0.$	(12)
ETD2	$y_{n+1} = y_n e^{ch} + h(g_0 f_n + g_1(f_n - f_{n-1}))$	(13)
ETD3	$y_{n+1} = y_n e^{ch} + h(g_0 f_n + g_1(f_n - f_{n-1}) + g_2(f_n - 2f_{n-1} + f_{n-2}))$	(14)
ETD4	$y_{n+1} = y_n e^{ch} + h(g_0 f_n + g_1(f_n - f_{n-1}) + g_2(f_n - 2f_{n-1} + f_{n-2}) + g_3(f_n - 3f_{n-1} + 3f_{n-2} - f_{n-3}))$	(15)

Table 1: ETD Schemes of order 1, 2, 3 and 4

where

$$g_0 = \frac{1}{ch}(e^{ch} - 1) \quad (16)$$

$$g_1 = \frac{1}{ch}(g_0 - 1) \quad (17)$$

$$g_2 = \frac{1}{ch}\left(g_1 + \frac{g_0}{2} - 1\right) \quad (18)$$

$$g_3 = \frac{1}{ch}\left(g_2 + \frac{g_1}{2} + \frac{g_0}{3} - 1\right) \quad (19)$$

LIMITING FORM OF THE EXPONENTIAL TIME DIFFERENCING

Start by considering

$$\lim_{c \rightarrow 0} g_0 = \lim_{c \rightarrow 0} \frac{1}{ch}(e^{ch} - 1). \quad (20)$$

from which we have

$$\begin{aligned} \lim_{c \rightarrow 0} g_0 &= \lim_{c \rightarrow 0} \frac{1}{ch} \left(\left(\sum_{n=0}^{\infty} \frac{(ch)^n}{n!} \right) - 1 \right) \\ &= \lim_{c \rightarrow 0} \frac{1}{ch} \left(\sum_{n=1}^{\infty} \frac{(ch)^n}{n!} \right) \\ &= \lim_{c \rightarrow 0} \left(\frac{1}{1!} + \sum_{n=2}^{\infty} \frac{(ch)^{n-1}}{n!} \right), \end{aligned}$$

and so

$$\lim_{c \rightarrow 0} g_0 = 1. \quad (21)$$

So that as $c \rightarrow 0$, ETD1 becomes

$$y_{n+1} = y_n + hf_n. \quad (22)$$

Equation (22) is the **Adams Bashforth of order 1** which can be found in Kreyszig (2006)

We find $\lim_{c \rightarrow 0} g_1$,

$$\lim_{c \rightarrow 0} g_1 = \lim_{c \rightarrow 0} \frac{1}{ch}(g_0 - 1), \quad (23)$$

$$\lim_{c \rightarrow 0} g_1 = \lim_{c \rightarrow 0} \frac{1}{ch} \left(\frac{1}{ch}(e^{ch} - 1) - 1 \right),$$

$$\begin{aligned}
&= \lim_{c \rightarrow 0} \frac{1}{(ch)^2} (e^{ch} - (1 + ch)), \\
&= \lim_{c \rightarrow 0} \frac{1}{(ch)^2} \left(\sum_{n=2}^{\infty} \frac{(ch)^n}{n!} \right), \\
&= \lim_{c \rightarrow 0} \sum_{n=2}^{\infty} \frac{(ch)^{n-2}}{n!}, \\
&= \lim_{c \rightarrow 0} \left(\frac{1}{2!} + \sum_{n=3}^{\infty} \frac{(ch)^{n-2}}{n!} \right), \\
&= \frac{1}{2!}
\end{aligned} \tag{24}$$

So that as $c \rightarrow 0$, ETD2 becomes

$$\begin{aligned}
y_{n+1} &= y_n + h \left((1)f_n + \left(\frac{1}{2!} \right) (f_n - f_{n-1}) \right) \\
y_{n+1} &= y_n + \frac{h}{2} (3f_n - f_{n-1})
\end{aligned} \tag{25}$$

Equation (25) is the **Adams Bashforth of order 2** in Kreyszig (2006).

For the $\lim_{c \rightarrow 0} g_2$,

$$\begin{aligned}
\lim_{c \rightarrow 0} g_2 &= \lim_{c \rightarrow 0} \frac{1}{ch} \left(g_1 + \frac{g_0}{2} - 1 \right), \\
&= \lim_{c \rightarrow 0} \frac{1}{ch} \left(\left(\frac{1}{2!} + \sum_{n=3}^{\infty} \frac{(ch)^{n-2}}{n!} \right) + \frac{1}{2} \left(\frac{1}{1!} + \sum_{n=2}^{\infty} \frac{(ch)^{n-1}}{n!} \right) - 1 \right) \\
&= \lim_{c \rightarrow 0} \frac{1}{ch} \left(\left(\frac{1}{2!} + \frac{1}{2} \left(\frac{1}{1!} \right) - 1 \right) + \sum_{n=3}^{\infty} \frac{(ch)^{n-2}}{n!} + \frac{1}{2} \sum_{n=2}^{\infty} \frac{(ch)^{n-1}}{n!} \right) \\
&= \lim_{c \rightarrow 0} \frac{1}{ch} \left(\sum_{n=3}^{\infty} \frac{(ch)^{n-2}}{n!} + \frac{1}{2} \sum_{n=2}^{\infty} \frac{(ch)^{n-1}}{n!} \right) \\
&= \lim_{c \rightarrow 0} \left(\sum_{n=3}^{\infty} \frac{(ch)^{n-3}}{n!} + \frac{1}{2} \sum_{n=2}^{\infty} \frac{(ch)^{n-2}}{n!} \right) \\
&= \lim_{c \rightarrow 0} \left(\frac{1}{3!} + \frac{1}{2} \left(\frac{1}{2!} \right) + \sum_{n=4}^{\infty} \frac{(ch)^{n-3}}{n!} + \frac{1}{2} \sum_{n=3}^{\infty} \frac{(ch)^{n-2}}{n!} \right) \\
&= \frac{5}{12}
\end{aligned} \tag{27}$$

So that as $c \rightarrow 0$, ETD3 becomes

$$\begin{aligned}
y_{n+1} &= y_n + h \left((1)f_n + \left(\frac{1}{2!} \right) (f_n - f_{n-1}) + \left(\frac{5}{12} \right) (f_n - 2f_{n-1} + f_{n-2}) \right) \\
y_{n+1} &= y_n + \frac{h}{12} (23f_n - 16f_{n-1} + 5f_{n-2})
\end{aligned} \tag{28}$$

Limiting Form of Exponential Time Differencing Schemes

Equation (28) is called the **Adams Bashforth of order 3** as in Kreyszig (2006).

For g_3 ,

$$\begin{aligned}
 \lim_{c \rightarrow 0} g_3 &= \lim_{c \rightarrow 0} \frac{1}{ch} \left(g_2 + \frac{g_1}{2} + \frac{g_0}{3} - 1 \right), & (29) \\
 &= \lim_{c \rightarrow 0} \frac{1}{ch} \left(\frac{1}{3!} + \frac{1}{2} \left(\frac{1}{2!} \right) + \sum_{n=4}^{\infty} \frac{(ch)^{n-3}}{n!} + \frac{1}{2} \sum_{n=3}^{\infty} \frac{(ch)^{n-2}}{n!} + \frac{1}{2} \left(\frac{1}{2!} + \sum_{n=3}^{\infty} \frac{(ch)^{n-2}}{n!} \right) \right. \\
 &\quad \left. + \frac{1}{3} \left(\frac{1}{1!} + \sum_{n=2}^{\infty} \frac{(ch)^{n-1}}{n!} \right) \right) \\
 &= \lim_{c \rightarrow 0} \frac{1}{ch} \left(-1 + \frac{1}{3!} + \frac{1}{2} \left(\frac{1}{2!} \right) + \frac{1}{2} \left(\frac{1}{2!} \right) + \frac{1}{3} \left(\frac{1}{1!} \right) + \sum_{n=4}^{\infty} \frac{(ch)^{n-3}}{n!} + \frac{1}{2} \sum_{n=3}^{\infty} \frac{(ch)^{n-2}}{n!} \right. \\
 &\quad \left. + \frac{1}{2} \sum_{n=3}^{\infty} \frac{(ch)^{n-2}}{n!} + \frac{1}{3} \sum_{n=2}^{\infty} \frac{(ch)^{n-1}}{n!} \right) \\
 &= \lim_{c \rightarrow 0} \frac{1}{ch} \left(\sum_{n=4}^{\infty} \frac{(ch)^{n-3}}{n!} + \sum_{n=3}^{\infty} \frac{(ch)^{n-2}}{n!} + \frac{1}{3} \sum_{n=2}^{\infty} \frac{(ch)^{n-1}}{n!} \right) \\
 &= \lim_{c \rightarrow 0} \left(\sum_{n=4}^{\infty} \frac{(ch)^{n-4}}{n!} + \sum_{n=3}^{\infty} \frac{(ch)^{n-3}}{n!} + \frac{1}{3} \sum_{n=2}^{\infty} \frac{(ch)^{n-2}}{n!} \right) \\
 &= \lim_{c \rightarrow 0} \left(\frac{1}{4!} + \frac{1}{3!} + \frac{1}{3} \left(\frac{1}{2!} \right) + \sum_{n=5}^{\infty} \frac{(ch)^{n-4}}{n!} + \sum_{n=4}^{\infty} \frac{(ch)^{n-3}}{n!} + \frac{1}{3} \sum_{n=3}^{\infty} \frac{(ch)^{n-2}}{n!} \right) \\
 &= \frac{3}{8} & (30)
 \end{aligned}$$

So that as $c \rightarrow 0$, ETD4 becomes

$$\begin{aligned}
 v_{n+1} &= y_n + h \left((1)f_n + \left(\frac{1}{2!} \right) (f_n - f_{n-1}) + \left(\frac{5}{12} \right) (f_n - 2f_{n-1} + f_{n-2}) \right. \\
 &\quad \left. + \frac{3}{8} (f_n - 3f_{n-1} + 3f_{n-2} - f_{n-3}) \right) \\
 v_{n+1} &= y_n + \frac{h}{24} (55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}) & (31)
 \end{aligned}$$

Equation (3.12) is the **Adams Bashforth of order 4** obtained in Kreyszig (2006).

NUMERICAL EXAMPLE

We shall consider the IVP

$$\frac{dy}{dx} = cy + xy^2, \quad y(0) = 1. & (32)$$

whose general solution is

$$y = \frac{c^2}{1 - cx + (c^2 - 1)e^{-cx}} & (33)$$

Tables 4.1 to 4.4 show the result with the choice of $c = 5$. As can be observed from the tables, the absolute errors propagated from the ETD methods is smaller than the corresponding AB method in each case. Hence, it becomes obvious that the ETD schemes are better than the AB schemes for stiff systems.

ETD1	AB1	EXACT	ErrorETD1	errorAB
1	1	1	0	0
1.0513	1.05	1.0513	0.0001	0.0013
1.1053	1.1026	1.1054	0.0001	0.0028
1.1622	1.158	1.1624	0.0002	0.0044
1.2222	1.2163	1.2225	0.0003	0.0062
1.2855	1.2777	1.2859	0.0004	0.0082
1.3522	1.3424	1.3528	0.0006	0.0104
1.4227	1.4106	1.4235	0.0008	0.0129
1.4971	1.4825	1.4981	0.001	0.0156
1.5757	1.5584	1.577	0.0013	0.0186
1.6588	1.6385	1.6604	0.0016	0.0219
1.7466	1.7231	1.7486	0.002	0.0255
1.8396	1.8125	1.8421	0.0025	0.0296
1.9381	1.9071	1.9411	0.003	0.034
2.0425	2.0072	2.0462	0.0037	0.039
2.1532	2.1132	2.1576	0.0044	0.0444
2.2707	2.2255	2.2761	0.0053	0.0505
2.3956	2.3447	2.402	0.0064	0.0573
2.5284	2.4713	2.5361	0.0077	0.0648
2.6699	2.6059	2.679	0.0091	0.0731

Table 2: Result from the execution of ETD1 with AB1

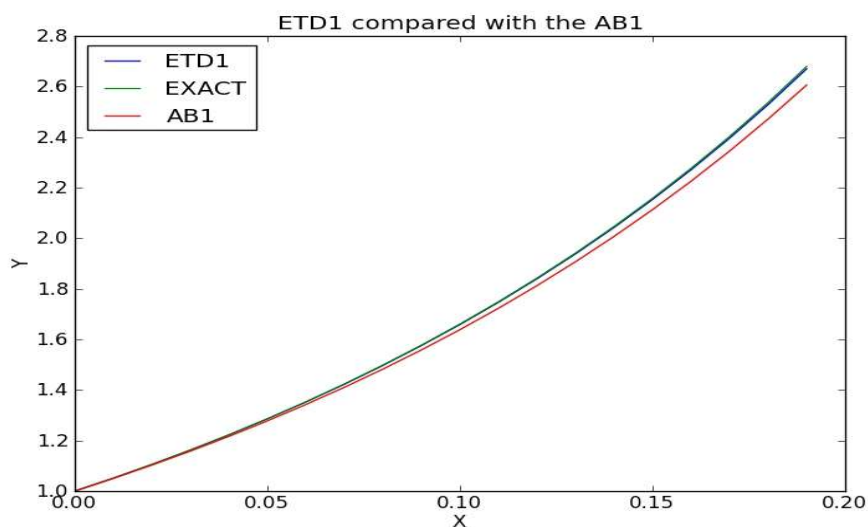


Figure 1: Graph of ETD1, AB1 and the exact solution, showing how far the AB1 deviates from exact solution compared with the ETD1 counterpart

Limiting Form of Exponential Time Differencing Schemes

ETD2	AB2	EXACT	errorETD	errorAB
1	1	1	0	0
1.0513	1.05	1.0513	0.0001	0.0013
1.1053	1.1039	1.1054	0.0001	0.0015
1.1623	1.1608	1.1624	0.0001	0.0016
1.2224	1.2207	1.2225	0.0001	0.0018
1.2858	1.2839	1.2859	0.0001	0.002
1.3527	1.3507	1.3528	0.0001	0.0022
1.4233	1.4211	1.4235	0.0002	0.0024
1.4979	1.4955	1.4981	0.0002	0.0026
1.5767	1.5741	1.577	0.0002	0.0029
1.6601	1.6572	1.6604	0.0003	0.0032
1.7483	1.7451	1.7486	0.0003	0.0035
1.8417	1.8383	1.8421	0.0004	0.0038
1.9407	1.9369	1.9411	0.0005	0.0042
2.0456	2.0415	2.0462	0.0006	0.0047
2.157	2.1525	2.1576	0.0007	0.0051
2.2753	2.2704	2.2761	0.0008	0.0056
2.4011	2.3958	2.402	0.0009	0.0062
2.535	2.5292	2.5361	0.0011	0.0069
2.6778	2.6714	2.679	0.0013	0.0076

Table 3: Result from the execution of ETD2 with AB2

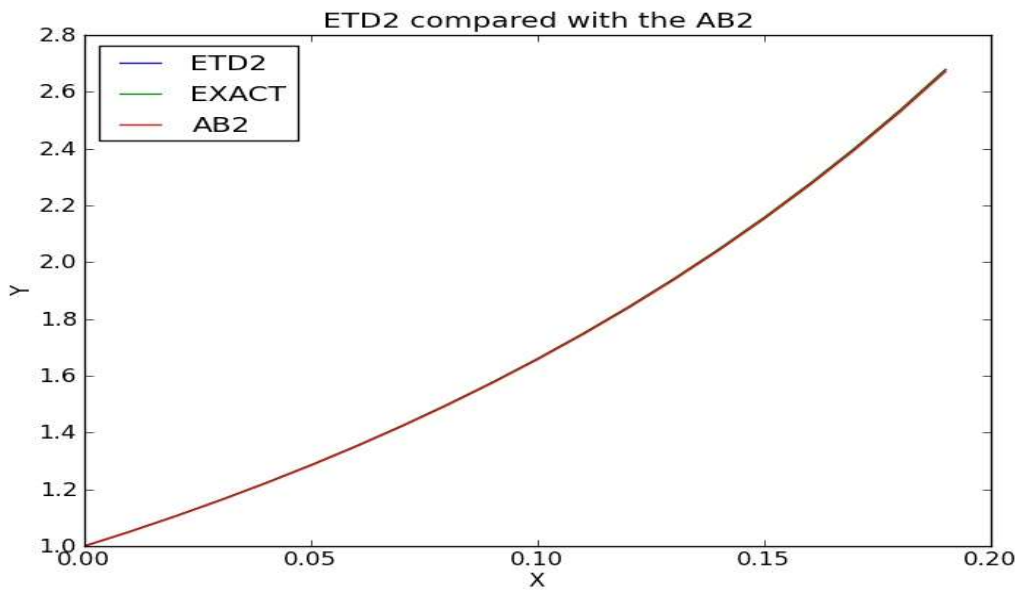


Figure 2: Graph of ETD2, AB2 and the exact solution. The errors are not very obvious from the graph but can be viewed from the table 3

ETD3	AB3	EXACT	errorET D	errorA B
1	1	1	0	0
1.0513	1.05	1.0513	0.0001	0.0013
1.1053	1.1026	1.1054	0.0001	0.0028
1.1623	1.1594	1.1624	0.0001	0.003
1.2224	1.2194	1.2225	0.0001	0.0031
1.2858	1.2826	1.2859	0.0001	0.0033
1.3527	1.3494	1.3528	0.0002	0.0035
1.4233	1.4198	1.4235	0.0002	0.0037
1.4979	1.4943	1.4981	0.0002	0.0039
1.5768	1.5729	1.577	0.0002	0.0041
1.6602	1.6561	1.6604	0.0002	0.0043
1.7484	1.7441	1.7486	0.0002	0.0046
1.8419	1.8373	1.8421	0.0002	0.0048
1.9409	1.936	1.9411	0.0003	0.0051
2.0459	2.0407	2.0462	0.0003	0.0054

Table 4: Result from the execution of ETD3 with AB3

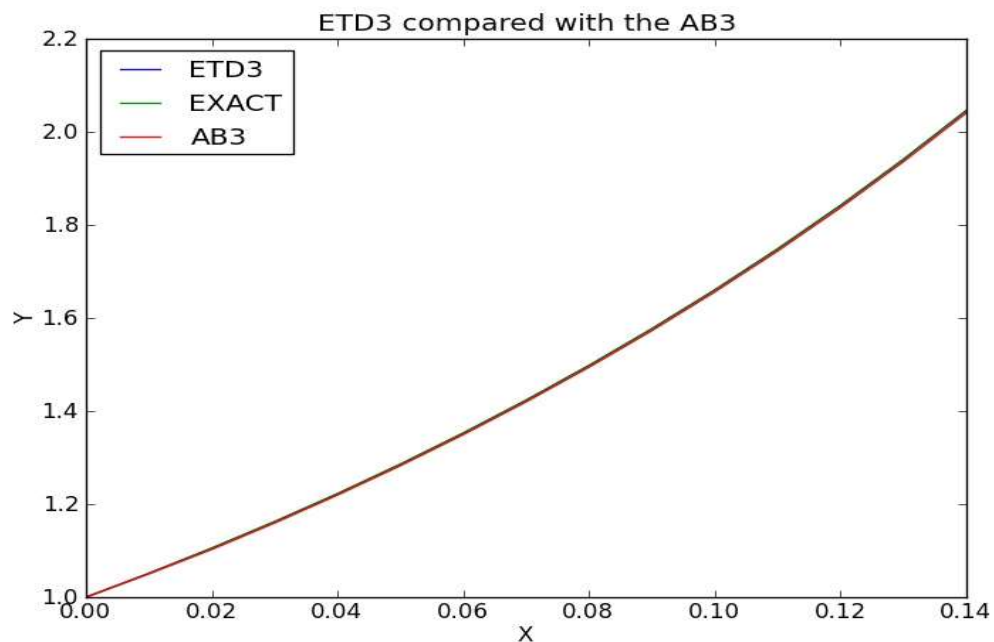


Figure 3: Graph of ETD3, AB3 and the exact solution. The errors are not very obvious from the graph but can be viewed from the table 4

Limiting Form of Exponential Time Differencing Schemes

ETD4	AB4	EXACT	errorETD	errorAB
1	1	1	0	0
1.0513	1.05	1.0513	0.0001	0.0013
1.1053	1.1026	1.1054	0.0001	0.0028
1.1621	1.158	1.1624	0.0003	0.0044
1.2222	1.2178	1.2225	0.0004	0.0047
1.2856	1.281	1.2859	0.0004	0.0049
1.3525	1.3477	1.3528	0.0004	0.0052
1.4231	1.418	1.4235	0.0004	0.0054
1.4977	1.4924	1.4981	0.0004	0.0057
1.5765	1.5709	1.577	0.0005	0.0061
1.6599	1.654	1.6604	0.0005	0.0064
1.7481	1.7419	1.7486	0.0005	0.0067
1.8416	1.835	1.8421	0.0005	0.0071
1.9406	1.9336	1.9411	0.0006	0.0075
2.0455	2.0382	2.0462	0.0006	0.0079
2.157	2.1492	2.1576	0.0006	0.0084
2.2754	2.2672	2.2761	0.0007	0.0089
2.4013	2.3926	2.402	0.0007	0.0094
2.5353	2.5261	2.5361	0.0008	0.01
2.6782	2.6684	2.679	0.0008	0.0106

Table 5: Result from the execution of ETD4 with AB4

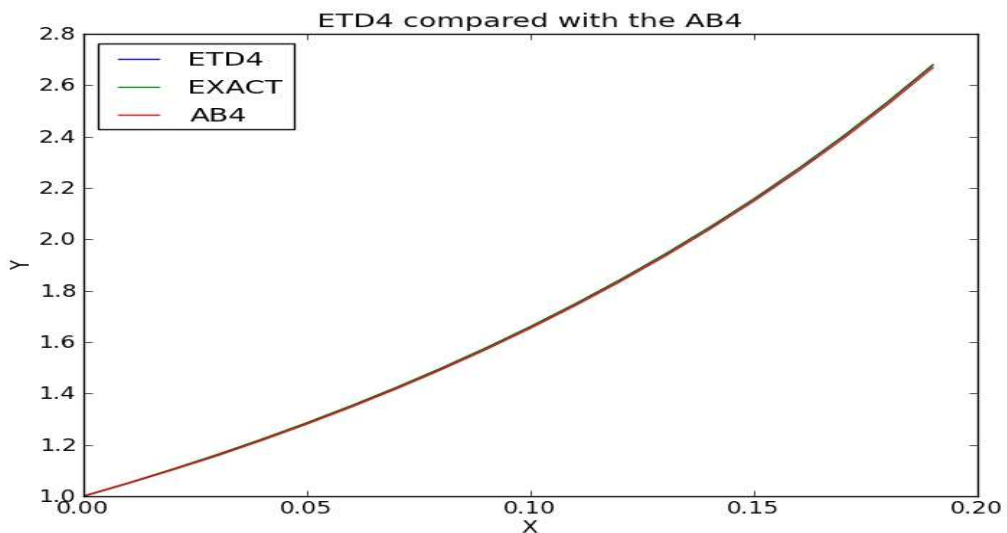


Figure 4: Graph of ETD4, AB4 and the exact solution. The errors are not very obvious from the graph but can be viewed from the table 5

.CONCLUSION

The exponential term in the ETD schemes pays for the stiffness and helps to recover terms that are about to be lost. The Exponential Time Differencing schemes of order 1, 2, 3 and 4 has been shown to reduce to Adams Bashforth (AB) schemes of the corresponding order as $c \rightarrow 0$.

These have been shown to be true for ETDs up to order 4 but a more rigorous proof will be needed to show that Exponential Time Differencing Scheme reduces to Adams Bashforth schemes for arbitrary order.

If we make the assumption from the start, that

$$\frac{dy}{dx} = f(x, y). \quad (34)$$

and we proceed in the same manner with which ETD was derived, we end up with the Adams Bashforth scheme. With this, one may want to jump to conclusion that the ETD reduces to AB as $c \rightarrow 0$ but it is derailing. Substituting $c = 0$ from the start makes the problem lose its ability to tackle stiff systems. Hence, it is important to prove from the result of [1] that the ETD will actually reduce to AB as $c = 0$

More so, the exponential term will be lost in the process, hence the essence of the ETD is defeated.

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Sons Inc.ISBN-13: 978-0-471-72897-9

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