



Nested Second Derivative General Linear Methods

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Abstract

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A family of nested second derivative general linear methods (NSDGLMs) which combines the advantages of linear multistep methods and Runge – Kutta methods is presented. The method possess high order with A-stability property, making it suitable for solving stiff initial value problems of ordinary differential equation. In implementation, the NSDGLM avoids the disadvantages of poor stability of linear multistep methods and high cost of implementation of Runge -Kutta methods. Numerical experiment was carried out on the Van der Pol equation, and it is observed that the numerical results of the NSDGLM compares with the reference solution from the MATLAB ode15s.

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INTRODUCTION

Differential equations are well known as essential tools in scientific modeling. In this paper, numerical integration of initial value problem (IVP) for systems of ordinary differential equations (ODEs) which we write in autonomous form as

$$\begin{cases} y'(x) = f(y(x)), & x \in [x_0, X] \\ y(x_0) = y_0 \in \mathbb{R}^m \end{cases} \quad (1)$$

where $f: \mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ is a given function satisfying some regularity conditions is presented.

The linear multistep methods (LMMs) and Runge-Kutta methods (RKMs) are well known methods for the numerical integration of ODEs. However, despite the efficiency of LMMs, they suffer from the Dahlquist's order barrier (Dahlquist, 1963). In the same vein, with RKMs having good stability properties (e.g A-stability, L-stability, or algebraic stability with high order), implicit RKMs requires very high cost of implementation (Huang, 2005).

Hybrid methods which are multistep methods with off-step points have been proposed in Gragg and Stetter (1964), Gear (1965), Butche and O'Sullivan (2002), and also in Ikhile and Okunoghae (2007) to mention a few, in order to overcome the disadvantages of LMMS while retaining certain characteristics of LMMs and sharing the property of utilizing data at off-step points like RKMs. Kulikov and Shindin (2007, 2009) considered the idea of nesting, where special implicit Runge-Kutta methods based on the Gauss quadrature formula was developed for stiff problems. Recently, Olatunji (2017), Olatunji and Ikhile (2017, 2018) considered families of second derivative multistep methods with nested hybrid evaluation, having good stability properties (A-stability) with high order.

Butcher (1966) introduced general linear methods (GLMs) as a unifying framework for studying stability, consistency and convergence for a wide

variety of methods. As in Butcher (2016), the general form of the GLM is given by

$$\begin{cases} Y_i = h \sum_{j=1}^s a_{ij} f(Y_j) + \sum_{j=1}^r u_{ij} y_j^{[n-1]}, & i = 1, 2, \dots, s \\ y_i^{[n]} = h \sum_{j=1}^s b_{ij} f(Y_j) + \sum_{j=1}^r v_{ij} y_j^{[n-1]}, & i = 1, 2, \dots, r \end{cases} \quad (2)$$

where $n = 1, 2, 3, \dots, N$, $x_{n+1} = x_n + h$ and Y_i approximates $y(x_n + c_i h)$; $i = 1, 2, \dots, s$. This can be written together as a partitioned $(s + r) \times (s + r)$ matrix, given as

$$\begin{bmatrix} Y \\ y^{[n]} \end{bmatrix} = \begin{bmatrix} A & U \\ B & V \end{bmatrix} \begin{bmatrix} hF(Y) \\ y^{[n-1]} \end{bmatrix} \quad (3)$$

where $Y, y^{[n]}, F(Y)$ is defined by

$$Y = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_s \end{bmatrix}; \quad F(Y) = \begin{bmatrix} f(Y_1) \\ f(Y_2) \\ \vdots \\ f(Y_s) \end{bmatrix}; \quad y^{[n]} = \begin{bmatrix} y_1^{[n]} \\ y_2^{[n]} \\ \vdots \\ y_r^{[n]} \end{bmatrix}$$

Second derivative general linear methods (SDGLM) were developed by Butcher and Hojjati (2005), where the class of methods possess Runge-kutta stability. Abdi and Hojjati (2011) proposed second derivative general linear methods (SDGLM) having the advantage of order and having A-stability properties. Also, Okuonghae and Ikhile (2014a) developed a second derivative RKM which has a simple transformation to the SDGLM. In Okuonghae and Ikhile (2014b), a new class of SDGLM having nearly almost Runge-Kutta stability was proposed. In what follows, we introduce nested SDGLM.

NESTED SDGLM

Consider the numerical integration of (1), by the second derivative general linear method defined by

$$\begin{bmatrix} Y \\ y^{[n]} \end{bmatrix} = \begin{bmatrix} A & \bar{A} & U \\ B & \bar{B} & V \end{bmatrix} \begin{bmatrix} hF \\ \frac{h^2 F''}{2} \\ y^{[n-1]} \end{bmatrix} \quad (4)$$

where $A, \bar{A}, B, \bar{B}, U, V$ are defined as

$$A = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & \beta_k^{(0)} \\ \beta_{v_0}^{(1)} & 0 & 0 & \dots & 0 & 0 & \beta_k^{(1)} \\ 0 & \beta_{v_1}^{(2)} & 0 & \dots & 0 & 0 & \beta_k^{(2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & \beta_k^{(l)} \\ 0 & 0 & 0 & \dots & \beta_{v_l}^{(l+1)} & 0 & \beta_k^{(l+1)} \\ 0 & 0 & 0 & \dots & 0 & \beta_{v_m}^{(k)} & \beta_k^{(k)} \end{bmatrix};$$

$$\bar{A} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & \lambda_k^{(0)} \\ 0 & 0 & 0 & \dots & 0 & 0 & \lambda_k^{(1)} \\ 0 & 0 & 0 & \dots & 0 & 0 & \lambda_k^{(2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & \lambda_k^{(l)} \\ 0 & 0 & 0 & \dots & 0 & 0 & \lambda_k^{(l+1)} \\ 0 & 0 & 0 & \dots & 0 & 0 & \lambda_k^{(k)} \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & \beta_{v_m}^{(k)} & \beta_k^{(k)} \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{bmatrix};$$

$$\bar{B} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & \lambda_k^{(k)} \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{bmatrix}$$

$$U = \begin{bmatrix} \alpha_{k-1}^{(0)} & \alpha_{k-2}^{(0)} & \dots & \alpha_2^{(0)} & \alpha_1^{(0)} \\ \alpha_{k-1}^{(1)} & \alpha_{k-2}^{(1)} & \dots & \alpha_2^{(1)} & \alpha_1^{(1)} \\ \alpha_{k-1}^{(2)} & \alpha_{k-2}^{(2)} & \dots & \alpha_2^{(2)} & \alpha_1^{(2)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_{k-1}^{(l)} & \alpha_{k-2}^{(l)} & \dots & \alpha_2^{(l)} & \alpha_{k-1}^{(l)} \\ \alpha_{k-1}^{(l+1)} & \alpha_{k-2}^{(l+1)} & \dots & \alpha_2^{(l+1)} & \alpha_{k-1}^{(l+1)} \\ \alpha_{k-1}^{(k)} & \alpha_{k-2}^{(k)} & \dots & \alpha_2^{(k)} & \alpha_{k-1}^{(k)} \end{bmatrix}$$

$$V = \begin{bmatrix} \alpha_{k-1}^{(k)} & \alpha_{k-2}^{(k)} & \dots & \alpha_2^{(k)} & \alpha_{k-1}^{(k)} \\ 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}$$

where

$$Y = \begin{bmatrix} y_{n+v_0} \\ y_{n+v_1} \\ y_{n+v_2} \\ \vdots \\ y_{n+v_l} \\ y_{n+v_m} \\ y_{n+k} \end{bmatrix}; \quad y^{[n]} = \begin{bmatrix} y_{n+k} \\ y_{n+k-1} \\ \vdots \\ y_{n+1} \end{bmatrix}; \quad y^{[n-1]} = \begin{bmatrix} y_{n+k-1} \\ y_{n+k-2} \\ \vdots \\ y_n \end{bmatrix}$$

$$F = \begin{bmatrix} hf_{n+v_0} \\ hf_{n+v_1} \\ hf_{n+v_2} \\ \vdots \\ hf_{n+v_l} \\ hf_{n+v_m} \\ hf_{n+k} \end{bmatrix}; \quad F' = F = \begin{bmatrix} h^2 f'_{n+v_0} \\ h^2 f'_{n+v_1} \\ h^2 f'_{n+v_2} \\ \vdots \\ h^2 f'_{n+v_m} \\ h^2 f'_{n+k} \end{bmatrix}$$

From the structure of the A matrix, it is seen that the implicitness is only on the output method (same as the last stage) thereby reducing computational cost. Thus, we call the method nested second derivative output method (NSDGLM).

ORDER AND STABILITY CONDITIONS

For simplicity, the method (4) can be expressed in stage values and output values for the step number n , given by

$$Y_i = h \sum_{j=1}^s a_{ij} f(Y_j) + h^2 \sum_{j=1}^s \bar{a}_{ij} f'(Y_j) + \sum_{j=1}^r u_{ij} y_j^{[n-1]}; \quad i = 1, 2, \dots, s$$

$$y_i^{[n]} = h \sum_{j=1}^s b_{ij} f(Y_j) + h^2 \sum_{j=1}^s \bar{b}_{ij} f'(Y_j) + \sum_{j=1}^r v_{ij} y_j^{[n-1]}; \quad i = 1, 2, \dots, r \quad (5)$$

where $s = k + 1$, and the $a_{ij}, \bar{a}_{ij}, u_{ij}, b_{ij}, \bar{b}_{ij}$ and v_{ij} are the entries of the matrices $A, \bar{A}, U, B, \bar{B}$ and V respectively.

Theorem 1. *The method (4) has order p if and only if*

$$e^{cz} = zAe^{cz} + z^2 \bar{A}e^{cz} + Uw + O(z^{p+1})$$

$$e^z = zBe^{cz} + z^2 \bar{B}e^{cz} + Vw + O(z^{p+1}) \quad (6)$$

where e^{cz} represents the vector with components given by $e^{ci z}; i = 1, 2, \dots, s$ and w represents the vector with entries given by

$$w_i = \sum_{m=0}^p \omega_{im} z^m; \quad i = 1, 2, \dots, r$$

Proof: The stage Y_j defined in the first equation of (5) can be written as

$$Y_i = y(x_n + c_i h) + O(h^{p+1})$$

$$= \sum_{m=0}^p \frac{C_i^m}{m!} y^{(m)}(x_n) h^m + O(h^{p+1})$$

It follows that

$$hf(Y_i) = hy'(x_n + c_i h) + O(h^{p+2})$$

$$= \sum_{m=1}^{p+1} \frac{C_i^{m-1}}{(m-1)!} y^{(m)}(x_n) h^m + O(h^{p+2})$$

$$= \sum_{m=1}^p \frac{C_i^{m-1}}{(m-1)!} y^{(m)}(x_n) h^m + O(h^{p+1})$$

and

$$h^2 f'(Y_i) = h^2 y''(x_n + c_i h) + O(h^{p+3})$$

$$= \sum_{m=2}^{p+2} \frac{C_i^{m-2}}{(m-2)!} y^{(m)}(x_n) h^m + O(h^{p+3})$$

$$= \sum_{m=2}^p \frac{C_i^{m-2}}{(m-2)!} y^{(m)}(x_n) h^m + O(h^{p+1})$$

The Taylor series expansion of the first step of the output method can be written in the form

$$y_i^{[1]} = \sum_{m=0}^p \left(\sum_{l=0}^m \frac{1}{l!} \omega_{i,m-l} \right) y^{(m)}(x_n) h^m + O(h^{p+1})$$

by this, equation (5), can be written as

$$\sum_{m=0}^p \left(C_i^m - \sum_{j=1}^s m a_{ij} C_j^{m-1} - \sum_{j=1}^s m(m-1) \bar{a}_{ij} C_j^{m-2} - m! \sum_{j=1}^r u_{ij} w_{jm} \right) \frac{h^m}{m!} y^{(m)}(x_n) = O(h^{p+1})$$

$$\sum_{m=0}^p \left(\sum_{l=0}^m \frac{1}{l!} \omega_{i,m-l} - \sum_{j=1}^s m b_{ij} C_j^{m-1} - \sum_{j=1}^s m(m-1) \bar{b}_{ij} C_j^{m-2} - m! \sum_{j=1}^r v_{ij} w_{jm} \right) \frac{h^m}{m!} y^{(m)}(x_n) = O(h^{p+1}) \quad (7)$$

Equating the coefficients of $\frac{h^m}{m!} y^{(m)}(x_n); m = 0(1)p$ in (7) to zero, then multiplying these coefficients by $\frac{z^m}{m!}$ and adding up from $m = 0$ to $m = p$, we obtain

$$e^{ci z} - z \sum_{i=1}^s a_{ij} e^{ci z} - z^2 \sum_{i=1}^s \bar{a}_{ij} e^{ci z} - \sum_{i=1}^r u_{ij} w_j = O(z^{p+1}) \quad i = 1, 2, \dots, s$$

$$e^z w_i - z \sum_{i=1}^s b_{ij} e^{ci z} - z^2 \sum_{i=1}^s \bar{b}_{ij} e^{ci z} - \sum_{i=1}^r v_{ij} w_j = O(z^{p+1}) \quad i = 1, 2, \dots, r \quad (8)$$

Hence, obtaining (6).

Theorem 2. Let $M(z)$ be the stability matrix of the NSDGLM defined by

$$M(z) = V + zB + z^2 \bar{B} (I - zA - z^2 \bar{A}) U \quad (9)$$

then when (5) is applied to the scalar test problem $y' = \lambda y$ and $z = \lambda h$ yields

$$y^{[n]} = M(z) y^{[n-1]} \quad (10)$$

Proof: Applying the method (5) on the scalar test problem $y' = \lambda y$, then $hF = h\lambda Y$ and $h^2F' = h^2\lambda^2 Y$ and replacing $z = \lambda h$ we obtain

$$\begin{bmatrix} Y \\ y^{[n]} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \bar{\mathbf{A}} & \mathbf{U} \\ \mathbf{B} & \bar{\mathbf{B}} & \mathbf{V} \end{bmatrix} \begin{bmatrix} -\frac{zY}{z^2Y} \\ y^{[n-1]} \end{bmatrix}$$

Thus, from the stages we have

$$Y = (zA + z^2\bar{A})Y + Uy^{[n-1]}$$

making Y subject and substituting into

$$y^{[n]} = (zB + z^2\bar{B})Y + Vy^{[n-1]}$$

and on solving yields

$$y^{[n]} = M(z)y^{[n-1]}$$

where

$$M(z) = V + (zB + z^2\bar{B})(I - zA - z^2\bar{A})U$$

and I is the unit matrix.

Definition 1. The stability polynomial of the NSDGLM $(\mathbf{A}, \bar{\mathbf{A}}, \mathbf{B}, \bar{\mathbf{B}}, \mathbf{U}, \mathbf{V})$ is defined by

$$\Pi(w, z) = \det(wI - M(z)) \quad (11)$$

Definition 2. The NSDGLM (4) is zero stable if the coefficient matrix V has exactly one eigenvalue on the unit circle (i.e. equal to 1).

Definition 3. The region of absolute stability of the NSDGLM (4) is the set

$$\varnothing = \{z \in C : |w_j| \leq 1, j = 1(1)k\} \quad (12)$$

i.e. if the root of $w_j, j = 0(1)k$ of (27) are less or equal to one in absolute value, such that those of magnitude one are not repeated.

Definition 4. The NSDGLM (4) is A -stable if the region of absolute stability lies in the entire left half of the z -plane (i.e. $z \in C^-$).

Definition 5. The NSDGLM (4) is $A(\alpha)$ -stable for some $\alpha \in [0, \frac{\pi}{2})$ if the wedge

$$S_\alpha = \{z : |Arg(-z)| < \alpha, z \neq 0\} \quad (13)$$

is contained in its region of absolute stability.

EXAMPLE METHODS

In this section, we give examples of NSDGLM discussed above. The construction of the methods is done using the order conditions in theorem 1.

Method of order $p = 3$

For the step number $k = 1$ the third order method obtained is given as

$$\begin{bmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & 0 & -\frac{3}{8} & 1 \\ \frac{4}{3} & -\frac{1}{3} & 0 & \frac{1}{6} & 1 \\ \frac{4}{3} & -\frac{1}{3} & 0 & \frac{1}{6} & 1 \end{bmatrix} \begin{bmatrix} hf_{n+\frac{1}{2}} \\ hf_{n+1} \\ \bar{h}^2 f'_{n+\frac{1}{2}} \\ h^2 f'_{n+1} \\ y_n \end{bmatrix} \quad (14)$$

The third order method has stability polynomial defined by

$$\Pi(w, z) = -1 + w - \frac{4z}{3} + \frac{wz}{3} - \frac{5wz^2}{6} + \frac{wz^3}{2} \quad (15)$$

The stability region for (14) in figure 1 shows that the third order method is A -stable.

Method of order $p = 4$

For the step number $k = 2$ the fourth order method obtained is given as

$$\begin{bmatrix} y_{n+\frac{1}{4}} \\ y_{n+\frac{3}{4}} \\ y_{n+2} \\ y_{n+2} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{39}{165} & 0 & 0 & -\frac{33}{128} & \frac{73}{65} & -\frac{9}{31} \\ \frac{165}{38} & 0 & -\frac{1407}{698} & 0 & 0 & \frac{47}{1216} & \frac{63}{698} & -\frac{9}{31} \\ 0 & \frac{32}{31} & -\frac{2}{31} & 0 & 0 & \frac{2}{31} & \frac{32}{31} & -\frac{1}{31} \\ 0 & \frac{32}{31} & -\frac{2}{31} & 0 & 0 & \frac{2}{31} & \frac{32}{31} & -\frac{1}{31} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} hf_{n+\frac{1}{4}} \\ hf_{n+\frac{3}{4}} \\ \bar{h}^2 f'_{n+\frac{1}{4}} \\ h^2 f'_{n+\frac{3}{4}} \\ h^2 f'_{n+2} \\ y_{n+1} \end{bmatrix} \quad (16)$$

with the stability matrix given as

$$M(z) = \begin{bmatrix} \frac{2(243z+2556z+7665z^2)}{4712+304z+10952z^2-10098z^3+3465z^4} & \frac{-2(76+124z+945z^2)}{(4712+304z+10952z^2-10098z^3+3465z^4)} \\ 1 & 0 \end{bmatrix}$$

the boundary locus plot of (16) shows also that the fourth order method is A-stable, as shown in figure 1. Therefore, higher order methods have been derived similarly using the stage values:

for 5th order method

$$c = \left(\frac{23}{8}, \frac{11}{4}, \frac{5}{2}, 3 \right)^T$$

For 6th order method

$$c = \left(\frac{63}{16}, \frac{31}{8}, \frac{15}{4}, \frac{7}{2}, 4 \right)^T$$

For 7th order method

$$c = \left(\frac{159}{32}, \frac{79}{16}, \frac{39}{8}, \frac{19}{4}, \frac{9}{2}, 5 \right)^T$$

The stability plot of the methods for k=1(1)5 (i.e p=3(1)7 respectively) is shown in figure 1 (indicating A-stability).

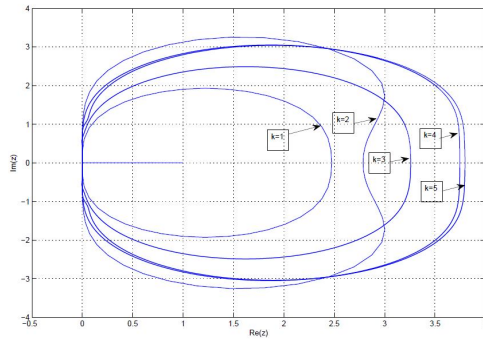


Figure 1: Stability plot of the NSDGLM for k=1(1)5.

IMPLEMENTATION OF THE NSDGLM

In this section, we discuss implementation procedure of the NSDGLM and thereafter implement the NSDGLM on some stiff problems of the form (1). Solving a stiff problem (1), the stages of the NSDGLM takes the form

$$Y = h\gamma f(Y) + h^2\bar{\gamma} \frac{\partial}{\partial y} f(Y) + Uy^{[n-1]}$$

Let $J(Y)$ denote the Jacobian matrix with entries given by

$$J(Y) = \frac{\partial}{\partial y_i} f_i(Y) \quad i = 1, 2, \dots, N$$

for a given N system of equations. Since the last stage method is implicit, the Newton's method

$$Y^{[n]} = Y^{[n-1]} + \Delta$$

where

$$\Delta = -\frac{F^{[n-1]}}{J(Y^{[n-1]})}$$

and

$$F^{[n-1]} = Y = h\gamma f(Y^{[n-1]}) + h^2\bar{\gamma} \frac{\partial}{\partial y} f(Y^{[n-1]}) + Uy^{[n-1]}$$

is required in other to resolve its implicitness. The starting predictor used for our method is the second derivative Euler scheme

$$y_{n+1} = y_n + hf_n + \frac{h^2}{2} f'_n$$

Implementing the NSDGLM in variable step size, we adopt the idea in Butcher (2016) to compute a new step size

$$h^* = 0.9 \left(\frac{TOL}{\bar{\omega}} \right)^{\left(\frac{1}{p} \right)} h$$

where, h is the last successful or failed step size, h^* the new step size to be used if h fails, TOL is the user's specified tolerance and $\bar{\omega}$ the estimated error at each step n .

Numerical experiments were carried out on the Van der Pol equation (Hairer and Wanner, 2010).

$$\begin{aligned} y_1'(x) &= y_2(x) & y_1(0) &= 2 \\ y_2'(x) &= ((1 - y_1^2(x))y_2(x) - y_1(x))/\epsilon & y_2(0) &= 0 \\ x &\in [0, 10] & \epsilon &= 10^{-4} \end{aligned}$$

The problem was solved using the fourth order ($p = 4$) NSDGLM and the fourth order second derivative linear multistep method (SDLMM) of

Enright (1974). These problem have been solved using the variable step size implementation, starting with the step size $h = 0.1$. Tables 1 and 2 shows the obtained results for the Van der Pol problem In the table, ns means number of steps carried out, nrs means number of rejected steps, nfe means number of function evaluations, h_e means last step size used, g_e means the global error and t_i means total time taken for computation. The

obtained results show that the NSDGLM outperforms the SDGLM. For all tolerances, the NSDGLM uses fewer computational time despite its more functional evaluation to that of the SDLMM. Moreover, SDLMM has larger error than that of the NSDGLM. The numerical results of the NSDGLM compares with the reference solution from MATLAB ode15s as seen from the graphs (figure 2 and 3).

Table 1: Numerical results of the Van der Pol equation solved by NSDGLM

TOL	ns	nrs	nfe	h_e	g_e	t_i
10^{-2}	2291	10	4582	2.5550×10^{-2}	2.1653×10^{-3}	0.05
10^{-4}	18656	8	37312	9.8529×10^{-4}	5.0133×10^{-5}	1.74
10^{-6}	186817	6	373634	1.6818×10^{-4}	5.2858×10^{-7}	467.51

Table 2: Numerical results of the Van der Pol equation solved by SDLMM (Enright, 1974)

TOL	ns	nrs	nfe	h_e	g_e	t_i
10^{-2}	1936	64	11936	5.1296×10^{-2}	7.6555×10^{-3}	0.09
10^{-4}	10184	18	22184	6.2658×10^{-4}	5.4774×10^{-5}	1.10
10^{-6}	202547	9	262547	5.6222×10^{-5}	5.3349×10^{-6}	510.91

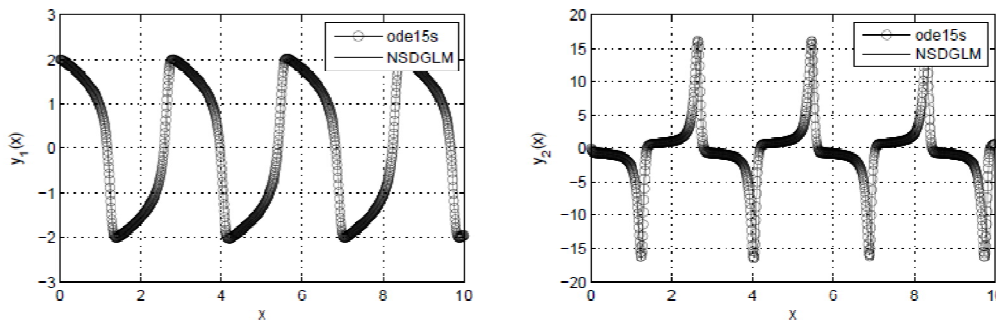


Figure 2: Graphical Solution of the Van der Pol equation

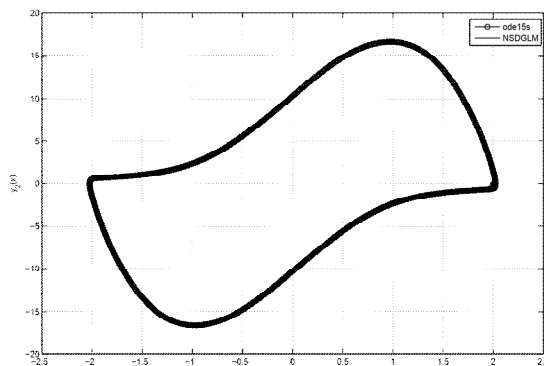


Figure 3: Phase plot of the Van der pol equation

CONCLUSION

Nested second derivative general linear methods has been presented. The order conditions and stability property (A - stability) of the method have been discussed. The methods constructed were A -stable up to order $p = 7$, making the NSDGLM suitable for solving stiff initial value problems of ordinary differential equations. The implementation procedure of the NSDGLM was also discussed and the fourth order method has been implemented using variable step size implementation on the Van der Pol equation. Future work will investigate at what step K , will instability set in.

REFERENCES

- Abdi A. and Hojjati G., (2011). *An extension of general linear methods*. Numer. Algorithm. Vol. 57, No. 2, 149–167.
- Butcher J.C., (1966). *On the convergence of the numerical solutions to ordinary differential equations*. Math. Comput. Vol. 20, 1–10.
- Butcher J.C., (2016) *Numerical methods for ordinary differential equivalent*, John Wiley & Sons, Ltd Chichester.
- Butcher J.C., and Hojjati G., (2005). *Second derivative methods with RK stability*, Numerical Algorithms. Vol. 40, 415-429.
- Butcher J. C. and O’Sullivan A. E., (2002). *Nordsieck methods with an off-step point*, Numerical Algorithms, Vol. 31, 87 – 101.
- Dahlquist G., (1963). *A special stability problem for Linear Multistep Methods*. BIT Vol. 3, 27–43.
- Enright W. H., (1974). *Second derivative multistep method for stiff ordinary differential equations*, SIAM J. Numerical Analysis. Vol. 11, No. 2, 321-331.
- Gear C.W., (1965). *Hybrid methods for initial problems in ODEs*. SIAM J. Numer. Anal. Vol. 2, 69–86.
- Gragg W.B., and Stetter H.J., (1964). *Generalized multistep predictor corrector methods*. J. Assoc. Comput. Mach. Vol. 11, 188–209.
- Hairer E. and Wanner G., (2010). *Solving ordinary differential equations II. Stiff and Differential – Algebraic problems Vol. II*, Springer-Verlag.
- Huang S. J. Y., (2005). *Implementation of General Linear Methods Stiff Ordinary Differential Equations*. Ph.D. Thesis, Department of Mathematics, University of Auckland.
- Ikhile M. N. O. and Okuonghae R. I., (2007). *Stiffly stable continuous extension of second derivative LMM with an off-step point for IVPs in ODEs*. J. Nig. Assoc. Math. Physics. Vol. 11, 175–190.
- Kulikov G. Yu and Shindin S. K., (2007). *Numerical Tests with Gauss-Type Nested Implicit Runge-Kutta Formulas*, ICCS, Part I, LNCS 4487, 136-143.
- Kulikov G. Yu and Shindin S. K., (2009). *Adaptive nested IRK formulas of Gauss Type*, Journal of Applied Numerical Mathematics, Vol. 59, 707-722.
- Okuonghae R. I., (2012). *A class of continuous hybrid LMM for stiff IVPs in ODEs*. Analele Stiintifice ale Universitat II Ali. Cuza Din Iasi (S. N.) MATEMATICA, Tomul LVIII, Vol. 2, 239–258.
- Okuonghae R. I. and Ikhile M. N. O., (2014a). *Second derivative general linear methods*. Numer Algor Vol. 67, No. 3, 637–654.
- Okuonghae R. I. and Ikhile M. N. O., (2014b). *Second derivative GLM with nearly ARK stability*. J. Numer. Math., Vol. 22, No. 2, 165–176.
- Olutunji P. O., (2017). *Second Derivative Multistep methods with Nested Hybrid Evaluation*, M.Sc. Thesis, Department of Mathematics, University of Benin, Benin City.
- Olutunji P. O. and Ikhile M. N. O., (2017). *Modified Backward Differentiation Formulas with Recursively Nested Hybrid Evaluation*. Journal of the Nigerian Association of Mathematical Physics, Vol. 40, 86-95.
- Olutunji P. O. and Ikhile M. N. O., (2018). *Second Derivative Multistep method with Nested Hybrid Evaluation*, Asian Research Journal of Mathematics, Vol.11, No. 4, 1 – 11.